

Minimal conditions for parametric continuity of a utility representation*

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Abstract

Dependence on the parameter is continuous when perturbations of the parameter preserves strict preference for one alternative over another. We characterise this property via a utility function over alternatives that depends continuously on the parameter. The class of parameter spaces where such a representation is guaranteed to exist is also identified. When the parameter is the type or belief of a player, these results have implications for Bayesian and psychological games. When alternatives are discrete, the representation is jointly continuous and an extension of Berge’s theorem of the maximum yields a continuous value function. We apply this result to generalise a standard consumer choice problem where parameters are price-wealth vectors. When the parameter space is lexicographically ordered, a novel application to reference-dependent preferences is possible.

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1 Introduction

It is often natural to assume that strict preference for one alternative over another is preserved when a parameter upon which preferences depend is perturbed. We refer to this property as *continuous parameter dependence* of preferences. The following quote provides the behavioural motivation for this assumption.

When processing sensory input, it is of vital importance for the neural systems to be able to discriminate a novel stimulus from the background of redundant, unimportant signals.

(Mejias et al. [25])

In other words, in the absence of robust preference, errors would compound, and we would be unable to perform many of our day-to-day activities. A similar point is made by [32], and these views may be traced back to von Neumann [40].

In the main theorem of this paper, we identify minimal conditions such that continuous parameter dependence is characterised by utility representation that is continuous in the parameter. The sense in which the conditions are minimal are as follows. First, the axioms on preferences are necessary and sufficient for the representation. Second, when the parameter space fails to satisfy the conditions of this theorem, there always exist preferences with no continuous representation even though they vary continuously with the parameter.

Throughout, we assume that the set of alternatives is countable and independent of the parameter space. Thus, no topological assumptions regarding the alternatives are necessary for our main theorem. However, when the set of alternatives is discrete (so that every element is isolated), our conditions

are also necessary and sufficient for the representation to be jointly continuous on pairs of alternatives and parameters. This ensures that, for the discrete case, our representation generalises existing results from the literature on jointly continuous utility [21, 4]. The connections with this literature are explored in subsection 3.2.

Joint continuity is useful for deriving a continuous value function (e.g. a continuous indirect utility function in consumer theory). In particular, it is a premise of Berge’s theorem of the maximum. This theorem provides sufficient conditions for such a value function to exist. To allow for alternatives to be discrete, we make a minor extension to Berge’s theorem so that the constraint set need only vary upper hemicontinuously over the parameter space. The result is a value function that is continuous and a choice correspondence that is upper hemicontinuous. To our knowledge, this result is novel and related to the literature on envelope theorems with discrete alternatives [28, 33].

We present three applications of the results. The first application is to a consumer demand setting with a parameter space that is augmented to allow for a general experimental setting that does not require the specification of a metric. The results provide a framework through which to formalise robust testing of empirical phenomena such as *preference reversals* [35, 38]: agents choose one alternative over another, and yet assign higher prices to the latter. The second application is to the literature on types in *Bayesian* and *psychological* games (Mertens and Zamir [26] and Geanakoplos, Pearce, and Stacchetti [9] respectively), where hierarchies of beliefs play a central role. The third application is to a simple two-alternative example of *reference dependent* preferences. The novelty is that the set of reference points are lexicographically ordered.

The next section introduces the model along with preliminary observations. The main theorem and related results appear in section 3. The applications are presented in section 4. Following the summary in section 5, the proof of the main theorem is presented in appendix A1. All remaining proofs appear in appendix A2.

2 Model

We begin with the basic model of preferences that depend on a parameter regardless of issues relating to continuity. Continuous parameter dependence and basic topological conditions on the parameter space are then described. Following this, we introduce and explore the key conditions on the parameter space that ensure a continuous representation exists.

2.1 Parameter dependent preferences

Let A denote a nonempty set of *alternatives*. Let X denote a nonempty set. We refer to an element of X as a *parameter*. Motivated by our interest in robust strict preference, we take statements of strict preference as primitive. For each x in X , $a <_x b$ denotes the statement “at x , alternative b is strictly preferred to alternative a ”. For each x in X , $<_x$ is the binary relation A that summarises all such preference statements. As such $<_x$ is a subset of $A \times A$, and is referred to as *preferences at x* or *given x* . For alternatives a and b such that neither $a <_x b$, nor $b <_x a$, we write $a \sim_x b$.

The shorthand $\{<_x\}_{x \in X}$ denotes the collection “ $<_x$ such that x belongs to X ”, and is the primitive object we refer to as *preferences*. Preferences are *parameter-free* whenever X is a singleton, otherwise, they are *parameter-dependent*. Thus, we do not require that each x determines a unique $<_x$.

The term *parameter dependence* will be used without reference to preferences when no possible confusion might arise. Similarly, we henceforth refer to “the decision maker” as Val.

Representing parameter dependence By a *representation* of preferences, we mean a function of the form $U : A \times X \rightarrow \mathbb{R}$ such that, for every x in X , and every $a, b \in A$, $a <_x b$ if and only if $U(a, x) < U(b, x)$. That is, for each x in X , there exists a *utility function* $u = U(\cdot, x) : A \rightarrow \mathbb{R}$ that represents preferences at x in the usual sense. By the properties of $<$ on \mathbb{R} , it is straightforward to show that if preferences have a representation, then they satisfy the following two conditions.

Axiom Asy. *If $a, b \in A$, then, for every $x \in X$, $a <_x b$ implies not $b <_x a$.*

Axiom NT. *If $a, b, c \in A$, then, for every $x \in X$, $a <_x b$ implies $c <_x b$ or $a <_x c$.*

In turn, when A is countable, *asymmetry* (Asy) and *negative transitivity* (NT) are standard sufficient conditions for the existence of a utility function at each $x \in X$. That is, sufficient for a representation of $\{<_x\}_{x \in X}$ in this case. With minor modifications, this result is due to Cantor [3]. More generally, (Asy) and (NT) are equivalent to assuming the weak preference relation $\lesssim_x = <_x \cup \sim_x$ is complete and transitive for every $x \in X$ (see [8]). Recall that \lesssim_x is complete if, for all a, b and c in A , $a \lesssim_x b$ or $b \lesssim_x a$ and \lesssim_x is transitive if $a \lesssim_x b \lesssim_x c$ implies $a \lesssim_x c$.

With a view to finding the weakest conditions for parametric continuity, we will typically assume that A is countable. This will be explicitly assumed in statements that follow.

2.2 Basic conditions on the parameter space

Recall that a neighbourhood of x is some subset $N \subseteq X$ such that $x \in N$ and $G \subseteq N$ for some open set G . The notion of an open set of X is only well-defined once a topology on X is identified. The topology on X provides a constraint on the perturbations that are allowed.

Recall that a *topology* on X is any collection τ of subsets G of X such that τ is closed under finite intersections and arbitrary unions. Usually we will suppress reference to τ and simply call X a topological space. Thus, by “ G is open” we mean $G \in \tau$ and by “ F is closed” we mean that for some $G \in \tau$, F is equal to the complement $X - G$ of G .

For certain results in the sequel, we will explicitly assume that: for each distinct $x, y \in X$, there exist disjoint open neighbourhoods N and M of x and y (i.e. X is a Hausdorff space). Throughout the sequel, we assume the more basic assumption that each singleton set $\{x\}$ is closed. One implication of this latter assumption is that the conditions we introduce in subsection 2.4 are sufficient for X to be Hausdorff.

2.3 Continuous parameter dependence

If preferences are such that, for every $a, b \in A$ such that $a <_x b$, there exists an open neighbourhood N of x in X such that $a <_y b$ for every other y in N , then we say that *parameter dependence is continuous at x* . Note that if $<_x = \emptyset$, this condition says nothing about preferences at x .

Suppose that parameter dependence is discontinuous at y . Then, for some $a, b \in A$ such that $a <_y b$ we have the following: for every neighbourhood N of y , there exists $x \in N$ such that $b \lesssim_x a$. (This follows directly from the definition of $<_x$ and \sim_x .) Thus, discontinuous parameter dependence at y

implies that the set $\{x : a <_x b\}$ fails to be open for some $a, b \in A$. On the other hand, when parameter dependence is continuous at x for every $x \in X$, the set $\{x : a <_x b\}$ is open in X for every $a, b \in A$.

Axiom CD. *Parameter dependence is continuous at x for every $x \in X$.*

Gilboa and Schmeidler [11, 10] also assume *continuous parameter dependence* (CD) in addition to completeness and transitivity of weak preference at each x . Whilst we identify minimal conditions for parametric continuity of the representation, Gilboa and Schmeidler impose further axioms and obtain a representation that is linear in the parameter.

Remark 1. *The map $x \mapsto <_x$ defines a correspondence on X with values in $2^{A \times A}$. This correspondence is lower hemicontinuous (l.h.c.) provided that, for every open $G \subseteq A \times A$, the set $\{x : <_x \cap G \neq \emptyset\}$ is open. Since this latter set is just the union of $\{x : a <_x b\}$ over the set $(a, b) \in G$, we see that (CD) implies $x \mapsto <_x$ is l.h.c. When A is discrete, so that every singleton in $A \times A$ is open, the converse is also true because $\{x : <_x \cap G \neq \emptyset\} = \{x : a <_x b\}$ when $G = \{(a, b)\}$.*

Characterising continuous parameter dependence For any function $U : A \times X \rightarrow \mathbb{R}$, we will say that U is *continuous at x* whenever the function $U(a, \cdot) : X \rightarrow \mathbb{R}$ is continuous at x for every $a \in A$.

Lemma 2. *If X is Hausdorff and the representation $U : A \times X \rightarrow \mathbb{R}$ of preferences is continuous at x , then parameter dependence is continuous at x .*

PROOF OF LEMMA 2. Let G be the set of points $x \in X$ such that parameter dependence is continuous at x , and let H be the set of points $x \in X$ such

that U is continuous at x . We will show that $H \subseteq G$. Suppose parameter dependence is discontinuous at x , so that $x \in X - G$. Then, for some $a, b \in A$ satisfying $a <_x b$, the following holds: for every open neighbourhood N of x , there exists $y \in N$ such that $b \lesssim_y a$. Consider the collection $\{N_\nu\}_{\nu \in D}$ of all neighbourhoods of x partially ordered by the inclusion (subset) relation \subseteq . Then we may take D to be a directed set that generates a net $(y_\nu)_{\nu \in D}$. This net is such that: $b \lesssim_{y_\nu} a$ for every $\nu \in D$; and it converges to x . This latter fact follows from our assumption that X is Hausdorff. Now since U is a representation, this means that $0 < U(a, x) - U(b, x)$ and for every ν , $U(a, y_\nu) - U(b, y_\nu) \leq 0$. Then zero is an upper bound for the latter set of points. Thus U is discontinuous at x , and so $x \in X - H$. \square

When U is continuous at x for every $x \in X$, we say U is *continuous in the parameter* or that it satisfies *parametric continuity*. Lemma 2 immediately implies that if the representation U satisfies parametric continuity, then (CD) holds. The following statement is significantly weaker than the converse of lemma 2, yet, even for two alternatives, it requires further conditions on the parameter space.

If (Asy), (NT) and (CD) hold, then preferences have a representation that is continuous in the parameter.

If this statement holds, then U *characterises* continuous parameter dependence. We now define and explore the minimal conditions on the parameter space such that it does.

2.4 Perfectly normal parameter spaces

A (topological) space X is *perfectly normal* if it is both normal and perfect. A space X is *normal*, if every pair of closed sets E, F can be separated. That

is, there exist open disjoint sets G and H in X such that $E \subseteq G$ and $F \subseteq H$. A space is *perfect* if every closed set is the intersection of countably many open sets in X .

The following example shows precisely how this property may be used to construct a utility representation for the case where there are just two alternatives. The main theorem extends this preliminary observation to the countable case using an equivalent definition of perfect normality (Michael's selection theorem) that we present below.

Example 3. Let $A = \{a, b\}$, and suppose both the sets $\{x : a <_x b\}$ and $\{x : b <_x a\}$ are nonempty. If Val's preferences satisfy (Asy), then these sets are disjoint. If Val's preferences satisfy (CD), then these sets are open and $F = \{x : b \lesssim_x a\}$ is closed.

If X is perfect, then since F is closed, there exists a countable collection $\{G_n\}_{n \in \mathbb{N}}$ of open sets satisfying $\bigcap_1^\infty G_n = F$. Note that $X - G_n \subset \{x : a <_x b\}$, so that F and $X - G_n$ are closed and disjoint. If X is normal, then the Urysohn lemma applies. This guarantees the existence of a continuous function $f_n : X \rightarrow \mathbb{R}$ such that $f_n(x) = 0$ on F and $f_n(x) = 1$ on $X - G_n$, and $0 \leq f_n(x) \leq 1$ otherwise. Now let $f = \sum_1^\infty 2^{-n} f_n$. As the uniform limit of continuous functions is continuous, f is continuous. Moreover, $f(x) > 0$ if and only if $a <_x b$. By the same argument, there exists another continuous nonnegative function $g : X \rightarrow \mathbb{R}$ such that $g^{-1}(0) = \{x : a \lesssim_x b\}$.

Let $U(a, x) = 0$ for each $x \in X$ and let

$$U(b, x) = \begin{cases} f(x) & \text{if } a <_x b, \\ -g(x) & \text{if } b <_x a, \\ 0 & \text{otherwise.} \end{cases}$$

The resulting function $U : A \times X \rightarrow \mathbb{R}$ is a utility representation at each x that characterises the continuous parameter dependence of Val's preferences.

Whilst the set-theoretic definition of a perfectly normal space is the most basic. For certain applications, other definitions are useful. To this end we introduce the following concept: $F \subseteq X$ is a *zero set* provided that $f^{-1}(0) = F$ for some continuous function $f : X \rightarrow \mathbb{R}$. Recall that, since $\{0\}$ is closed in \mathbb{R} , for any continuous $f : X \rightarrow \mathbb{R}$, the set $F = f^{-1}(0)$ is closed in S . That is, the zero sets are always closed. The converse is only true when X is perfectly normal.

Definition 4. *X is perfectly normal if and only if every closed subset of X is a zero set.*

A third, equivalent definition of perfect normality is provided by the following restatement of Michael's selection theorem [27, Theorem 3.1'''].

Theorem (Good and Stares [12]). *X is perfectly normal if and only if whenever $g, h : X \rightarrow \mathbb{R}$ are upper and lower semi-continuous respectively and $g \leq h$, then there is a continuous $f : X \rightarrow \mathbb{R}$ such that $g \leq f \leq h$ and $g(x) < f(x) < h(x)$ whenever $g(x) < h(x)$.*

Examples of perfectly normal spaces It is not hard to see that a metrizable space is perfectly normal. In example 3, for instance, let G_n consist of points that are of distance at most n^{-1} from F . Indeed, by considering the usual metric $|\cdot|$ on the nonnegative real numbers \mathbb{R}_+ , the latter is metrizable and hence perfectly normal.

An example of a set that is perfectly normal and compact, but not metrizable is developed in the context of an application to reference dependence in subsection 4.3. This is the product $[0, 1] \times_{\text{lex}} \{0, 1\}$ of the unit interval $[0, 1]$ with the two-element, discrete set $\{0, 1\}$ with topology generated by the lexicographic order (where the first dimension is dominant). It is sometimes referred to the *split interval*.

The split interval is important, because all compact, separable ordered spaces are order isomorphic to one of its subsets [30]. This ensures that, in the order topology, every such space is homeomorphic to a subset of the split interval. Recall that two sets are *homeomorphic* whenever there is a continuous bijection (isomorphism) with continuous inverse between them.

Another space that is homeomorphic to the split interval is the set \mathbb{F} of increasing functions on $[0, 1]$ with values in $\{0, 1\}$ with the topology of pointwise convergence [39].¹ As a result, the latter is compact and perfectly normal, but not metrizable. As we show in the sequel, these examples of 4.3 serve to distinguish the present model from its complement.

What if the space fails to be perfectly normal? Whilst axiom (CD) is often well motivated in applications, it is natural to question the importance of representing continuous parameter dependence. That is to say: what if preferences satisfy (Asy), (NT) and (CD), but there is no representation that is continuous in the parameter? The following example shows how it may arise when X is not perfectly normal.

Example 5. *Let $A = \{a, b\}$ and suppose Val's preferences $\{\prec_x\}_{x \in X}$ satisfy (Asy), (CD), and that $F = \{x : a \sim_x b\}$ for some closed set F . Suppose that F is not a zero set. That is, there is no continuous, real-valued g on X such that $g^{-1}(0) = F$. (See 4.3 for an explicit example of such a set.) Since preferences satisfy (Asy), there exists a representation of preferences. Let $U : A \times X \rightarrow \mathbb{R}$ be any such representation. Let $f = U(a, \cdot) - U(b, \cdot)$. Then since U is a representation, $f^{-1}(0) = F$. This implies that $U(a, \cdot) - U(b, \cdot)$ is discontinuous. By the algebra of continuous functions, at least one of $U(a, \cdot)$*

¹In this case, the topology of pointwise convergence is the collection of all unions of finite intersections of sets $W(x, G) = \{f \in \mathbb{F} : f(r) \in G\}$, where $r \in [0, 1]$ and $G \subseteq \{0, 1\}$.

and $U(b, \cdot)$ is discontinuous.

The issues that arises when, as in example 5, the representation is discontinuous and preferences satisfy (CD), are as follows. First, a comparative statics exercise is difficult, for the modeller cannot rely on closeness to the boundary of $\{x : a <_x b\}$ being associated with eventual closeness of $U(a, \cdot) - U(b, \cdot)$ to zero. (Limits can no longer be relied upon for derivatives for instance.) The second arises in computational settings, where the utility function is used in the place of preferences and the modeller needs to approximate the underlying parameter space. The approximation is more costly if false conclusions such as $a <_x b$ for some $x \in F$ are to be avoided. This situation does not arise when the space is perfectly normal.

Counterexamples of perfectly normal spaces There are many kinds of parameter space that fail to be perfectly normal. These are useful in delineating the scope of the main theorem of this paper. Some of these, like the lexicographic unit square $[0, 1] \times_{\text{lex}} [0, 1]$, are discussed in subsection 4.3 arise when the parameter space is not perfect. Others arise when the space is not normal. Salient examples are provided by the usual *product topology* and often arise in the setting where the parameter space consists of beliefs (probability measures).² We discuss some of these in subsection 4.2.

For now, a simple example where perfect normality fails, is the product topology on the set $\{0, 1\}^S$ of functions on an arbitrary uncountable set S with values in $\{0, 1\}$. An element of this set might be interpreted as an uncountable “sequence” of coin tosses or, following Savage [34], as a function on states into the outcome space $\{0, 1\}$.

²Recall that for topological space X and Y , the product topology is the smallest topology such that, for any G open in X and H open in Y , the set $G \times H$ is open in $X \times Y$.

Perfectly normal spaces of beliefs In some of the applications that appear in section 4, we will let the parameter space be a set of probability distributions on a state space S . For this purpose, we need the following partial extension of [31, Theorem 6.4]. The relevant definitions appear in remark 7 below.

Proposition 6. *Let S be a compact set of states. Let Σ be the smallest σ -algebra containing all the zero sets of S , and let $\Delta(S)$ be the set of probability measures on Σ endowed with the weak* topology.*

1. *$\Delta(S)$ is metrizable if and only if S is metrizable.*
2. *If $\Delta(S)$ is perfectly normal, then so is S .*

Part 2 of proposition 6 is proved in appendix A2. With minor modifications, part 1 follows from [31, Theorem 6.4] and the connection between the Baire σ -algebra and the Borel σ -algebra that we describe below. Missing from the above result is the converse implication: S perfectly normal implies $\Delta(S)$ perfectly normal. Nonetheless, for our present purposes, an important fact that proposition 6 reveals is the following:

if S not perfectly normal, then neither is $\Delta(S)$.

When S is compact, Σ of proposition 6 is usually referred to as the Baire σ -algebra (the Baire sets). Note that it is more common to let Σ be the smallest σ -algebra that contains every closed set of S (the Borel sets). But recall that when S is perfectly normal, every closed subset is a zero set. Thus, when S is compact and perfectly normal, the Borel sets coincide with the Baire sets. In contrast, when S is not perfectly normal, there exists some closed subset F of S that is not a zero set. In this case, although every Baire set is

a Borel set, the converse does not hold. This is precisely when proposition 6 matters, for then the set of beliefs $\Delta(S)$ is not perfectly normal.

Remark 7. *When S is finite, Σ is usually the collection of all subsets of S (the discrete topology on S) and $\Delta(S)$ is then identified with the simplex in \mathbb{R}^S with the usual product topology. When the set of states is infinite, it is common to require that the set of states S is measurable and endowed with a σ -algebra Σ of subsets. (Σ can be any collection of subsets that is closed under complementation, countable intersection and countable union.) When considering the set of probability measures $\Delta(S)$ on S , it is common to consider the weak* topology on $\Delta(S)$. The weak* topology is defined as follows. Let $C(S)$ be the set of real-valued, continuous functions on S . Let Σ be a σ -algebra on S . The weak* topology on the set of countably additive probability measures $\Delta(S)$ on (S, Σ) is the smallest topology on $\Delta(S)$ such that, for each $f \in C(S)$, the linear functional $\mu \mapsto \int_S f \, d\mu$ is continuous on $\Delta(S)$.*

3 Results

First, we characterise the axioms on preferences of the preceding section through a function that is both a utility at each parameter and continuous across parameters. A corollary to the main theorem then characterises perfectly normal parameter spaces. Connections with results from the literature on jointly continuous representations appear in subsection 3.2. For A discrete, a jointly continuous representation on $A \times X$ is possible. In subsection 3.3, we use this joint continuity to derive a value function that is continuous in the parameter. This requires an extension of Berge's theorem of the maximum and leads to a choice function that is upper hemicontinuous in the

parameter.

3.1 Main theorem

The following result is equivalent to the classic representation of a single binary relation by Cantor [3] in the case that X is a singleton.³

Theorem 8. *For A countable and X perfectly normal, (Asy), (NT) and (CD) hold if and only if there exists a function $U : A \times X \rightarrow \mathbb{R}$ such that for every a in A and x in X ,*

1. $U(\cdot, x)$ is a utility representation of $<_x$; and
2. $U(a, \cdot)$ is a continuous function on X .

The proof of theorem 8 appears in appendix A1.

The requirement that X is perfectly normal is essential for the sufficiency of the axioms in theorem 8. If X is not a perfectly normal space, there are (possibly many) preferences that satisfy continuous parameter dependence, but have no representation that is continuous in the parameter. The following corollary formalises this statement and its converse.

Corollary 9. *X is a perfectly normal if and only if every $\{<_x\}_{x \in X}$ that satisfies (Asy), (NT) and (CD) on a countable set A has a representation that is continuous in the parameter.*

PROOF OF COROLLARY 9. If X is perfectly normal, then theorem 8 completes the sufficiency argument. With minor modifications, the converse follows by example 5.

³In fact, Cantor's theorem holds for the case where indifference sets are singletons. This latter distinction is minor when \sim_x is an equivalence relation, for Cantor's theorem then applies to the quotient set A/\sim_x .

□

3.2 Joint continuity for discrete alternatives

It is often useful to require that the representation of preferences is jointly continuous on $A \times X$. Examples include consumer demand theory (see subsection 4.1), game theory and the formalisation of prospect theory by [19]. The literature on jointly continuous representations of preferences $\{\prec_x\}_{x \in X}$ includes [17], [15] and [22], [21] and, more recently, [4]. For an introductory survey see [24]. Levin [21, Theorem 1] provides the simplest and most easily comparable continuity condition for a jointly continuous representation. Instead of (CD), Levin assumes

Axiom JC. *The set $\{(x, a, b) : a \precsim_x b\}$ is closed in $X \times A \times A$.*

Levin's theorem also requires that X is metrizable and A is a countable union of compact sets. Note that the set $\{(x, a, b) : a \precsim_x b\}$ of (JC) is the graph of $x \mapsto \precsim_x$. When $A \times A$ is compact Hausdorff, (JC) implies that the correspondence $x \mapsto \precsim_x$ is upper hemicontinuous [16, Theorem 2.2.3]. *Upper hemicontinuity* (u.h.c.) holds if, for every closed $F \subseteq A \times A$, the set $\{x : \precsim_x \cap F \neq \emptyset\}$ is closed in X .

Example 10. *Let A be a compact and infinite set and let $F = \{a\} \times A$. Then (JC) requires that*

$$\{x : \precsim_x \cap F \neq \emptyset\} = \bigcup_{b \in A} \{x : a \precsim_x b\}$$

is closed in X . In contrast, (CD) is only equivalent to assuming $\{x : a \precsim_x b\}$ is closed in X for every $a, b \in A$. In particular, it does not imply this infinite union of closed sets is closed.

Example 10 shows that, in general, (CD) is weaker than (JC). However, when A is discrete, the following corollary shows that theorem 8 yields a jointly continuous representation.

Corollary 11. *Let A be discrete. If the function $U : A \times X \rightarrow \mathbb{R}$ satisfies condition 2 of theorem 8, then it is jointly continuous.*

The proof of corollary 11 appears in appendix A2. This shows that, when A is discrete and countable, theorem 8 is a generalisation of both Levin [21, Theorem 1] and Caterino, Ceppitelli, and Maccarino [4, Theorems 4.1 and 4.2]. The reason is straightforward: these theorems require that X is either metrizable or submetrizable, whereas we only require that it is perfectly normal.

X is *submetrizable* if there exists a metric space Y and a continuous bijection $f : X \rightarrow Y$. This definition allows the inverse function f^{-1} to be discontinuous. Taking X to be the space \mathbb{F} , or the split interval yields a perfectly normal parameter space that is not submetrizable [2]. Another example of a perfectly normal, but not submetrizable space is the long line. This is the noncompact lexicographic product $[0, \omega_1) \times_{\text{lex}} [0, 1)$, where $[0, \omega_1)$ is the set of ordinal numbers that are countable and ω_1 is the first uncountable ordinal. This demonstrates the existence of applications to which theorem 8 applies, but the other results do not: even when A has just two alternatives.

If, on the otherhand, X is not perfectly normal, then corollary 9 guarantees the existence of preferences that fail to satisfy condition 2 of theorem 8. Since this is necessary for a joint continuity of the representation, the latter also fails to hold for such preferences. Note that this latter argument holds regardless of the cardinality and topology of A .

3.3 Maximum theorem for discrete alternatives

Joint continuity of $U : A \times X \rightarrow \mathbb{R}$ is the first premise of Berge's theorem of the maximum. Berge's theorem provides sufficient conditions for U to give rise to a continuous value function $V : X \rightarrow \mathbb{R}$. V is the function that selects the supremum of the values $U(\cdot, x)$ takes for each $x \in X$. The same conditions in Berge's theorem also give rise to a u.h.c. (see previous subsection) optimal choice correspondence $\mathcal{C} : X \rightarrow 2^A - \emptyset$.

The remaining premises of Berge's theorem relate to the set of constraints on A that Val faces at each $x \in X$. Thus, if $\mathcal{F}(x)$ denotes the set of alternatives available to Val at $x \in X$, then Berge's theorem requires that $\mathcal{F} : X \rightarrow 2^A - \emptyset$ is compact-valued, u.h.c., and l.h.c.⁴ \mathcal{F} is *compact-valued* if $\mathcal{F}(x)$ is compact for each $x \in X$. When \mathcal{F} is both u.h.c. and l.h.c., it is said to be continuous.

When A is discrete, the requirement that \mathcal{F} is continuous is often too strong. The following lemma illustrates the problem.

Lemma 12. *Let $A = \{a, b\}$ and let $Y = [0, 1]$. Suppose that $\mathcal{F} : Y \rightarrow 2^A - \emptyset$ is continuous. Then \mathcal{F} is constant.*

The proof of lemma 12 appears in appendix A2. When \mathcal{F} is constant, any application to the usual constrained maximisation problems of consumer and producer theory is precluded, for the budget set cannot vary with prices. Fortunately, when A is discrete, we only require that \mathcal{F} is u.h.c.

Theorem 13. *For A discrete and countable, and X perfectly normal, (Asy), (NT) and (CD) hold if and only if there exists a jointly continuous repre-*

⁴Recall from subsection 2.3 that \mathcal{F} is *lower hemicontinuous* (l.h.c.) if, for every open $G \subseteq A$, $\{x : \mathcal{F}(x) \cap G \neq \emptyset\}$ is open in X .

sentation $U : A \times X \rightarrow \mathbb{R}$ of $\{<_x\}_{x \in X}$. For any $\mathcal{F} : X \rightarrow 2^A - \emptyset$ that is u.h.c. and compact-valued, we have

1. $V(\cdot) := \max \{U(a, \cdot) : a \in \mathcal{F}(\cdot)\}$ is a continuous on X ;
2. $\mathcal{C}(\cdot) := \operatorname{argmax} \{U(a, \cdot) : a \in \mathcal{F}(\cdot)\}$ is a u.h.c. on X .

The proof of theorem 13 appears in appendix A2. The necessity of the axioms in theorem 13 is useful for many applied settings where modellers simply posit a jointly continuous utility function. It means that they assume (CD) holds.

4 Applications

In each of the following four subsections, we apply the results of section 3. The first application is to consumer theory where parameters are the product of a standard set of price-wealth vectors with another set of parameters. The second application is to the literature on “topologies on types” and psychological games. The third application is to belief dependence in a finance setting. The final application is to a setting where preferences exhibit reference dependence and the parameter space is a lexicographically ordered set.

4.1 Consumer theory with discrete commodities

A common setting where continuous parameter dependence is assumed is the theory of consumer demand. Here Berge’s theorem of the maximum yields an indirect utility function that continuous in the parameter and a demand correspondence that is u.h.c. (see subsection 3.3) in the parameter. (See

[36] for a detailed exposition). This is usually a stepping stone to envelope theorems and other tools for comparative static analysis.

Let $A \subset \mathbb{R}_+^{n-1}$ be a countable and discrete set of commodities. This might, for instance, be the set \mathbb{Z}_+^{n-1} of vectors with nonnegative integer-valued entries, or with rational entries that have decimal expansions restricted to (at most) 10 decimal places. Whilst this assumption is atypical, it has received relatively recent attention in [33, 28]. It has also been motivated in the game theoretic setting by [10, 1] and is standard in empirical settings where discrete choice econometric models (see [23]) are often used.

We assume X is the cartesian product of the set price-wealth vectors \mathbb{R}_{++}^n with some other set of parameters Θ . The latter may include other socio-economic data in econometric settings or characterise frames in experiments.

Lemma 14. *If Θ is perfectly normal, then so is $\mathbb{R}_{++}^n \times \Theta$.*

PROOF OF LEMMA 14. By [37, p.249], the cartesian product of a second countable space with a perfectly normal space is perfectly normal. \mathbb{R} is second countable as it has a countable basis: the open intervals with rational endpoints. \square

Each element of X is denoted by $x = (p, w, \theta)$, where p is the vector of prices (p_1, \dots, p_{n-1}) and w denotes wealth. Val's ability to choose elements of A is constrained by her budget. The budget correspondence varies with the parameter in the following way:

$$\mathcal{B} : X \rightarrow 2^A - \emptyset, \quad x \mapsto \{a \in A : p \cdot a \leq w\}. \quad (*)$$

With a view to ensuring the existence of a maximal element, we assume $\mathcal{B}(x)$ is compact for each x . Since A is discrete, this holds if and only if $\mathcal{B}(x)$ is finite for each $x \in X$.

Lemma 15. \mathcal{B} is u.h.c.

PROOF OF LEMMA 15. Since \mathcal{B} is independent of $\theta \in \Theta$, it suffices to consider sequences in \mathbb{R}_{++}^n . We prove that \mathcal{B} satisfies the following definition for upper hemicontinuity: for any sequence (p^k, w^k) in \mathbb{R}_{++}^n with limit (p, w) and open $G \subseteq A$ such that $\mathcal{B}(p, w) \subseteq G$, there exists $l \in \mathbb{N}$ such that for all $k \geq l$, $\mathcal{B}(p^k, w^k) \subseteq G$.⁵

Since A is discrete, $\mathcal{B}(p, w)$ is open and it suffices to prove the case where $G = \mathcal{B}(p, w)$. Since $\mathcal{B}(p, w)$ is finite, there exists a minimal $\delta > 0$ such that for all $a \in A - \mathcal{B}(p, w)$, $\|a - b\| > \delta$ for every $b \in \mathcal{B}(p, w)$.

Seeking a contradiction, we suppose there exists a sequence $(p^k, w^k) \rightarrow (p, w)$ with the following property: for every $l \in \mathbb{N}$, there exists $k \geq l$ such that $r_k = p^k \cdot a^k - w^k \leq 0$ and $s_k = p \cdot a^k - w > 0$. Passing to this subsequence, note that since $(p^k, w^k) \rightarrow (p, w)$, $r_k - s_k \rightarrow 0$. Then $r_k \leq 0 < s_k$ implies $s_k \rightarrow 0$. Fix $\epsilon > 0$, then there are infinitely many k such that $0 < s_k < \epsilon$. By assumption, the set $\mathcal{B}(p, w + \epsilon)$ is finite and contains every element in $\mathcal{B}(p, w)$ and every a^k that defines s_k . This ensures that the sequence (s_k) is finite-valued. But since $s_k \rightarrow 0$, there exists $l \in \mathbb{N}$ such that for every $k \geq l$, $s_k = 0$. This contradiction completes the proof. \square

Preferences For each $x \in X$, we assume Val is able to rank the elements of A according to $<_x$ with a view to identifying the best element(s) in $\mathcal{B}(x)$. Thus $\{<_x\}_{x \in X}$ satisfies (Asy) and (NT). This yields a representation $U : A \times X \rightarrow \mathbb{R}$ satisfying condition 1 of theorem 8. If Val is indifferent between two or more best elements, all such elements belong to her demand correspondence at x . The latter is a map $\mathcal{D} : X \rightarrow 2^A - \emptyset$ such that $\mathcal{D}(x) \subseteq \mathcal{B}(x)$ for all $x \in X$. The standard model assumes that Θ is a single-

⁵This definition is equivalent to the earlier definition by [16, Lemma 2.1.1].

ton and that, for all $x, y \in X$, $<_x = <_y$. The most natural generalisation would let preferences vary across Θ . For the present purposes, the only additional assumption we require is that (CD) holds.

Continuous parameter dependence For the purposes of conducting a comparative statics analysis a minimal requirement is that there exists a continuous indirect utility function $V : X \rightarrow \mathbb{R}$, $x \mapsto \max\{U(a, x) : a \in \mathcal{B}(x)\}$, and that \mathcal{D} is u.h.c. The latter ensures that the demand correspondence is continuous whenever it is a function. Since we have assumed that A is countable and discrete, X is perfectly normal, \mathcal{B} is u.h.c. and compact-valued and preferences satisfy (Asy), (NT) and (CD), theorem 13 yields the desired properties for V and \mathcal{D} .

4.2 Topologies on types

The present results are important for the literature on topologies on types [26, 7, 5]. Mertens and Zamir [26] assume only that S is compact. The authors interpret a point in S as a “full listing of the strategy spaces and payoff functions [of the players in a game]”. They then define a hierarchy of beliefs to be a sequence $\Delta_0, \Delta_1, \Delta_2, \dots$ such that Δ_0 is a compact subset of S , and for each $k > 0$, Δ_k is a compact subset of $\Delta_{k-1} \times [\Delta(\Delta_{k-1})]^n$. Since S need not be perfectly normal, and moreover, the product of perfectly normal spaces is not, in general, perfectly normal, proposition 6 in conjunction with corollary 9 tell us that the utility functions allowed by the model may not reflect the underlying parametric continuity of preference. [7, 5] consider the special case where S is finite and there are two players. The metrics they identify on the space of types ensure the resulting type space is perfectly normal.

The literature on psychological games constructs a similar hierarchy of beliefs. Our alternatives correspond to their outcomes, and our parameters correspond to their beliefs. (Geanakoplos, Pearce, and Stacchetti [9, p.65] state that “payoffs for player i are defined first on the outcomes (given any belief profile b) and only afterward extended ...”.) Similar to [7, 5], [9] assumes the basic set S on which beliefs are constructed is finite. They are therefore able to obtain a metrizable hierarchy of beliefs. In a related paper, beginning with preferences Gilboa and Schmeidler [10] provide a decision theoretic version with no explicit hierarchy of beliefs. As we have pointed out above, their axioms are a superset of ours. Their assumptions also ensure the set of beliefs is metrizable.

4.3 Lexicographic reference dependence

This subsection develops a detailed example showing that the results apply to problems that cannot be modelled using either a parameter-free utility function or the pre-existing results in the literature on jointly continuous utility representations. To fix ideas, we develop the example within a framework of reference-dependent preferences, as in [19]. The distinctive feature of our model is that the parameter space is the lexicographically ordered set $[0, 1] \times_{\text{lex}} \{0, 1\}$ that was introduced in section 2 as the split interval. We then extend the example to provide another where there is no continuous representation even though preferences satisfy (CD).

Our choice of parameter space is important, for it is a leading example of a compact space that is perfectly normal, but not metrizable. Since it is also homeomorphic to the space \mathbb{F} of *increasing* functions on $[0, 1]$ with values in $\{0, 1\}$ that was introduced in subsection 2.4, the results also apply to the setting where \mathbb{F} is the set of parameters.

Reference dependent preferences Let $A = \{a, b\}$ be the set of prospects (alternatives), and let $X = [0, 1] \times \{0, 1\}$ be the set of reference points (parameters). Assume that (Asy) holds, and that $\{x : a \sim_x b\} = I \times \{0, 1\}$ for some closed and nondegenerate interval $I \subseteq [0, 1]$. As in example 3, (NT) trivially holds since $|A| = 2$.

Order topology on reference points Recall the lexicographic ordering $<_{\text{lex}}$ over X that ranks $x \in X$ higher than $y \in X$ if and only $y_1 < x_1$ or $[y_1 = x_1 \text{ and } y_2 < x_2]$. The ordering $<_{\text{lex}}$ need not reflect Val's "preferences" over reference points in general, and in any case, it is distinct from her preferences $\{<_x\}_{x \in X}$ on A .

For any $y, z \in X$, each of $\{x : y <_{\text{lex}} x\}$ and $\{x : x <_{\text{lex}} z\}$ is an open order interval of X . From these basic sets, we derive a topology τ by taking unions of finite intersections of such intervals generates the lexicographic order topology on X . Because of the discrete nature of the second dimension of X , an arbitrarily small open neighbourhood of a point $y = (y_1, 1)$ is of the form $\{x : y \leq_{\text{lex}} x <_{\text{lex}} z\}$, for some z satisfying $y_1 < z_1$. This neighbourhood is of the form

$$\left((y_1, z_1) \times \{0, 1\} \right) \cup \{y\} \quad (**)$$

Similarly, when $y_2 = 0$, small enough perturbations consider $x \leq_{\text{lex}} y$, so that $x_1 < y_1$. (See [39] for more on this.)

Continuous reference dependence The collection of sets τ is distinct from the one obtained by considering X as a subspace of \mathbb{R}^2 with the usual, Euclidean topology. It is also distinct from the topology obtained by considering X as a subspace of the \mathbb{R}_+^2 with the lexicographic order topology [29, p.107]). Nonetheless, it may be reasonable to suppose that Val's preferences

on A satisfy (CD) with respect to τ . (CD) holds if the sets $\{x : a <_x b\}$ and $\{x : b <_x a\}$ are members of τ . The following lemma confirms this is the case.

Lemma 16. *Preferences $\{<_x\}_{x \in X}$ satisfy (CD).*

PROOF OF LEMMA 16. We show that set $\{x : a \sim_x b\} = I \times \{0, 1\}$ is closed in X . By [14, Proposition 2.1] the set $I \times_{\text{lex}} \{0, 1\}$ is homeomorphic to $X = [0, 1] \times_{\text{lex}} \{0, 1\}$ for every closed and nondegenerate $I \subseteq [0, 1]$. \square

Representation of preferences With the topology τ , the set X is a well known example of a perfectly normal topological space that is not metrizable [13]. Thus, by theorem 8, there is a representation of Val's preferences that characterises continuous parameter dependence. By continuous parameter dependence, in this setting, we mean that for every $x \in X$ such that $a <_x b$, there exists an open order interval of $<_{\text{lex}}$ of the form $(**)$ containing x such that $a <_y b$ for every y in that interval. That is, a function $U : A \times X \rightarrow \mathbb{R}$ such that, for each $x \in X$, $U(\cdot, x)$ is a utility function on A and, for each $a \in A$, $U(a, \cdot)$ is continuous on X .

Other approaches do not apply Since X is not (sub)metrizable, other results from the literature on jointly continuous representations that were discussed in subsection 3.2 do not apply. Furthermore, the fact that $(x_1, 0) <_{\text{lex}} (x_1, 1)$ for each $x_1 \in [0, 1]$ means that there is no real-valued utility representation of $<_{\text{lex}}$. (Any such representation would have uncountably many open gaps [6].) In turn, this implies that there is no utility representation of any preference that Val might have over the yet larger set $A \times X$.

Generalising the example By theorem 8, the above example can immediately be extended to a countable number of alternatives A . Somewhat surprisingly, the next proposition confirms that the present example cannot be extended so that the second dimension contains any other elements $0 < x_2 < 1$. Any such parameter space (including the full unit square $[0, 1]^2$) fails to be perfectly normal, so that, by corollary 9, continuous reference dependence may not be characterised by a continuous function. For the purposes of the following proposition, let $Y = [0, 1] \times_{\text{lex}} \{0, \frac{1}{2}, 1\}$ and recall that I is any nondegenerate, closed interval of $[0, 1]$.

Proposition 17. *There is no continuous function $U : A \times Y \rightarrow \mathbb{R}$ such that $U(a, x) - U(b, x) = 0$ if and only if $x \in I \times \{0, 1\}$.*

Proposition 17 is proved in appendix A2. Hernández-Gutiérrez [14, Proposition 2.1] strengthens this result: any closed subset of $[0, 1] \times \{0, 1\}$ that has no isolated points is homeomorphic to $I \times \{0, 1\}$. This shows that there is a large class of parameter dependent preferences that do not have a continuous representation.

5 Summary

We have given conditions on preferences and the parameter space for a general model of parametric continuity of preference. The main theorem shows that preferences satisfying the axioms can be represented by a function that is a utility given the parameter and is continuous on the parameter space.

Whilst the main drawback of the present model is that the set of alternatives must be countable, this assumption has allowed us to obtain the minimal conditions for parametric continuity. Firstly, the axioms on preferences

are necessary and sufficient for parametric continuity of the representation. Secondly, if the parameter space is not perfectly normal, then there exist preferences that vary continuously with the parameter, but have representation that is continuous in the parameter.

When the set of alternatives is discrete, (CD), the axiom that captures continuous parameter dependence, is both necessary and sufficient for joint continuity of the representation on the product of alternatives and parameters. This yields a generalisation of existing results from the literature on jointly continuous representations to the case where the parameter space is perfectly normal. Via a simple extension of Berge's theorem of the maximum, this joint continuity allowed us to derive (i) a value function that is continuous and (ii) a choice correspondence that is upper hemicontinuous.

The applications demonstrated that there are novel settings to which the present results appear uniquely suited.

inline (??)

A1 Proof of the main theorem

The necessity of axioms (Asy) and (NT) for part 1, is implied by classical (parameter-free) representation theorems for each x . Lemma 2 confirms that (CD) is necessary.

Sufficiency of the axioms Let $\{1, 2, 3 \dots\}$ be an arbitrary enumeration of A , and by $[j]$ we will denote the subset of A that contains the first j elements of the enumeration. By $U^j : [j] \times X \rightarrow \mathbb{R}$ we will denote the utility representation of the projection of preferences $\{<_x : x \in X\}$ onto the first j elements of the enumeration. That is, if we recall that for each $x \in X$, $<_x$ is

a subset of $A \times A$, then we see that $\{<_x : x \in X\} \subset (A \times A)^X$. Hence by the projection of preferences onto $[j]$ we mean $\{<_x : x \in X\} \cap ([j] \times [j])^X$. (This is a well defined intersection since $([j] \times [j])^X \cap (A \times A)^X = ([j] \times [j])^X$.)

We use this projection to proceed by induction on A . For the basic case, let $U^1(1, x) = 0$ for all $x \in X$. By condition (Asy), U^1 is a representation for the projection of preferences onto $[1] \times [1]$ and it is clearly continuous. This completes the proof for the basic case. The induction hypothesis is the following. Suppose that for some $j \geq 1$, there exists a representation U^{j-1} of the projection of preferences onto $[j-1]$. From this we obtain a representation of the projection onto $[j]$.

For $a \in [j-1]$ let $U^j(a, \cdot) = U^{j-1}(a, \cdot)$. By the induction hypothesis, for all $a, b \in [j-1]$ and $x \in X$ we have,

$$a <_x b \iff U^j(a, x) < U^j(b, x),$$

and on $[j-1]$, U^j is continuous. To complete the inductive step, we must select a continuous function $U^j(j, \cdot)$ on X such that for each x , $U^j(\cdot, x) : [j] \rightarrow \mathbb{R}$ represents $<_x \cap ([j] \times [j])$.

Summary of inductive step Define upper and lower envelopes, g and h respectively, of $U^j([j-1], X)$ relative to alternative j . Check this pair of functions satisfy the conditions for Michael's selection theorem (the version by [12] that was defined in section 2). First, $g : X \rightarrow \mathbb{R}$ is weakly dominated by $h : X \rightarrow \mathbb{R}$ pointwise; second, they are equal if and only if for some k in $[j-1]$, $j \sim_x k$; third they are respectively upper and lower semi-continuous. This, together with the fact that X is perfectly normal, implies, via Michael's selection theorem, that the required function $U^j(j, \cdot)$ exists.

Definition of upper and lower envelopes The following step is commonly taken in the construction of envelopes. We introduce two fictional alternatives \underline{a} and \bar{a} , such that for all $x \in X$ and $k \in [j]$, we have $\underline{a} <_x k <_x \bar{a}$. Accordingly, we define $[j - 1]' := [j - 1] \cup \{\underline{a}, \bar{a}\}$, and for each $x \in X$, let $U^j(\underline{a}, x) = -\infty$ and $U^j(\bar{a}, x) = +\infty$. Both are clearly continuous functions from X to the extended real line and Michael's selection theorem ensures that $U^j(j, x) = \pm\infty$ only if $U^j(\underline{a}, x) = U^j(\bar{a}, x)$, and this is clearly impossible. Moreover, for all $x \in X$, there exists $k, l \in [j - 1]'$ such that $k \lesssim_x j$ and $j \lesssim_x l$. Thus, the following are well defined:

$$\begin{aligned} g(x) &:= \max \{U^j(k, x) : k \lesssim_x j \text{ and } k \in [j - 1]'\}, \\ h(x) &:= \min \{U^j(k, x) : j \lesssim_x k \text{ and } k \in [j - 1]'\}. \end{aligned}$$

Applying Michael's selection theorem The following three lemmata ensure that g and h satisfy the conditions of g and h respectively in Michael's selection theorem.

Lemma 18. *For all $x \in X$, $g(x) \leq h(x)$.*

PROOF OF LEMMA 18. On the contrary, suppose that for some $x \in X$, $h(x) < g(x)$. Then, by construction, there exists $k, l \in [j - 1]$ such that $k \lesssim_x j$, $j \lesssim_x l$. But since

$$h(x) := U^j(l, x) < U^j(k, x) =: g(x),$$

we either have a violation of (NT), or a violation of the induction hypothesis (that $U^{j-1}(\cdot, x)$ was order-preserving at each x on $[j - 1]$). \square

Lemma 19. *For all $x \in X$: $g(x) = h(x)$ iff $k \sim_x j$ for some $k \in [j - 1]$.*

PROOF OF LEMMA 19. If $g(x) = h(x)$, then, by construction, there is some $k \in \{l : l \lesssim_x j\} \cap \{l : j \lesssim_x l\}$, and by (Asy), for every l in the intersection of these sets $l \sim_x j$. Conversely, if $j \sim_x k$, then both $k \lesssim_x j$ and $j \lesssim_x k$. \square

Lemma 20. $g : X \rightarrow \mathbb{R}$ is upper semicontinuous.

A symmetric argument to the one that follows, but with inequalities and direction of weak preference reversed, shows that h is lower semicontinuous.

PROOF OF LEMMA 20. Recall (or see [18, p.101]) that g is upper semicontinuous provided the set $\{x : r \leq g(x)\}$ is closed for each $r \in \mathbb{R}$. Note that by construction of g and the definition of maximum,

$$\{x : r \leq g(x)\} = \bigcup_{k \in [j-1]'} (\{x : r \leq U^j(k, x)\} \cap \{x : k \lesssim_x j\}).$$

As finite union of closed sets is closed. The following arguments complete the proof: firstly $U^j(k, \cdot)$ is continuous, so that $\{x : r \leq U^j(k, x)\}$ is closed (preimage of a closed set is closed); and secondly, $\{x : k \lesssim_x j\}$ is closed by (CD). \square

The countably infinite case The above argument holds for each j in \mathbb{N} .⁶ For countably infinite A , we choose $U : A \times X \rightarrow \mathbb{R}$ such that its graph satisfies $\text{gr } U = \bigcup_{j \in \mathbb{N}} \text{gr } U^j$. Since Michael's selection theorem is used at each j , for this step we appeal to the axiom of dependent choice. Alternatively, following [20, p.23], let $U(j, \cdot) = U^j(j, \cdot)$ for each $j \in \mathbb{N}$, and again appeal to the axiom of (dependent) choice.

⁶I thank Atsushi Kajii for bringing this subtle issue to my attention.

A2 Remaining proofs

PROOF OF PART 2 OF PROPOSITION 6. For each $s \in S$, the Dirac measure δ_s on Σ is the function that assigns value 1 to any set that contains s and is zero otherwise. The preimage of the open set $\{r \in \mathbb{R} : r \neq 1\}$ under δ_s is equal to the union of all sets that do not contain s ; it is therefore equal to $S - \{s\}$, and is therefore open. Thus, the mapping $s \mapsto \delta_s$ is thereby continuous and injective. Recall that every continuous injection that is a closed map is also an embedding. That is, a homeomorphism onto its image. Thus, if the image $\delta_S \subset X$ of S under δ is closed, and we will have shown that there exists a subspace of X that is not perfectly normal. This implies that X itself fails to be perfectly normal, for every subspace of a perfectly normal space inherits the same property.

The following argument shows that δ_S is indeed closed. Consider any net $\{\delta_s\}$ in δ_S converging weakly to μ in $\mathcal{P}(S)$. By the definition of weak convergence, $\int_S f d\delta_s \rightarrow \int_S f d\mu$, for each continuous $f : S \rightarrow \mathbb{R}$. Now $\int_S f d\delta_s = (f_*\delta_s)(\mathbb{R}) = f(s)$ for each s , and since f is continuous and S is compact, $f(s)$ converges to some k in the image of f . Hence, $\int_S f d\mu = k$ and μ lies in $\delta(S)$ and the proof is complete. \square

PROOF OF COROLLARY 11. Fix $(a, x) \in A \times X$ and consider, for some directed set D , a net $E = ((a_\nu, x_\nu))_{\nu \in D}$ in $A \times X$ with limit (a, x) . We show that $U(a_\nu, x_\nu) \rightarrow U(a, x)$. Recall that (a, x) is the limit of E if and only if, for every neighborhood N of (a, x) , there exists $\mu \in D$ such that for every $\nu \geq \mu$, $(a_\nu, x_\nu) \in N$. Since A is discrete, $\{a\}$ is open and for some N_x open in X , the set $\{a\} \times N_x$ is an (open) neighborhood of (a, x) in the product topology on $A \times X$. Thus, there exists μ such that for every $\nu \geq \mu$, $U(a_\nu, x_\nu) = U(a, x_\nu)$. Finally, condition 2 of theorem 8 ensures that $U(a, x_\nu) \rightarrow U(a, x)$. \square

PROOF OF LEMMA 12. Suppose otherwise. In particular suppose that for some $x \in [0, 1]$, $\mathcal{F}(x) = B$ and for some $y > x$, $\mathcal{F}(y) \neq B$.

The first case is $B = A$. Then since $\mathcal{F}(y) \neq \emptyset$, without loss of generality, suppose $\mathcal{F}(y) = \{a\}$ and let $G := \{a\}$. Let $F^+(G)$ denote $\{z : \mathcal{F}(z) \subset G\}$ and let $F^-(G)$ denote $\{x : \mathcal{F}(x) \cap G \neq \emptyset\}$. If $F^+(G)$ is open, then \mathcal{F} is u.h.c. However, by Ichiishi [16, p.32], $F^+(G) = X - F^-(A - G)$, so that $F^-(A - G) = F^-(\{b\})$ is closed. It is also nonempty, for $x \in F^-(\{b\})$ and not equal to $[0, 1]$ as it does not contain y . But then it fails to be open, a contradiction of the assumption that \mathcal{F} is l.h.c. A similar argument shows that u.h.c. fails whenever l.h.c. holds.

The second case is where $B \neq A$ for all $z \in [0, 1]$. In this case, the same argument shows that u.h.c. and l.h.c. cannot simultaneously hold. \square

PROOF OF THEOREM 13. First note that, since A is discrete, $\mathcal{F}(x)$ is compact iff $\mathcal{F}(x)$ is finite. For part 1, the proof of Ichiishi [16] requires upper semicontinuity of $(a, x) \mapsto U(a, x)$ and u.h.c. of $\mathcal{F}(\cdot)$. By corollary 11, U is jointly continuous, and so the proof of part 1 follows from that of part 2.

Since A is discrete, it is metrizable. Since $\mathcal{C} = \mathcal{C} \cap \mathcal{F}$, and \mathcal{F} is u.s.c. and compact valued, part 2 follows from [16, Lemma 2.2.2], provided the graph of \mathcal{C} is closed. That is, provided the set $\text{gr } \mathcal{C} = \{(x, a) \in X \times A : a \in \mathcal{C}(x)\}$ is closed in $X \times A$. For any directed set D , let $(x_\nu, a_\nu)_{\nu \in D}$ be any net with values in $\text{gr } \mathcal{C}$ and limit equal to (\bar{x}, \bar{a}) . Since A is discrete, the singleton set $\{a\}$ is the smallest open neighbourhood of any $a \in A$. Thus $(x_\nu, a_\nu)_{\nu \in D}$ satisfies the property that for some $\mu \in N$, $a_\nu = \bar{a}$ for all $\nu \geq \mu$. Thus, (x_ν, a_ν) is eventually in $G \times \{\bar{a}\}$ for some open $G \subset X$. Since $\mathcal{C}(x_\nu) \subseteq \mathcal{F}(x_\nu)$ for every $\nu \in D$, $\bar{a} \in \mathcal{F}(x_\nu)$ for every $\nu \geq \mu$. Since \mathcal{F} is u.s.c., its graph is closed, and $(\bar{x}, \bar{a}) \in \text{gr } \mathcal{F}$. Then by joint continuity of U , $\lim_\nu U(x_\nu, a_\nu) = U(\bar{x}, \bar{a})$. Now suppose that there exists $a \in A$ such that $U(\bar{x}, \bar{a}) < U(\bar{x}, a)$, so that \bar{a} is

not an argmax at \bar{x} . But this would contradict the assumption that \bar{a} is an argmax for all $\nu \geq \mu$. Thus $(\bar{x}, \bar{a}) \in \text{gr } C$, as required. \square

PROOF OF PROPOSITION 17. Recall that an arbitrary continuous function $f : X \rightarrow \mathbb{R}$ satisfies the property that $G_n = \{x : |f(x)| < \frac{1}{n}\}$ is open for each $n \in \mathbb{Z}_{++}$. We show that $\bigcap_1^\infty G_n \neq F$. The proof is immediate unless $F \subseteq G_n$ for each $n \in \mathbb{Z}_{++}$, so suppose this case holds. Then, by [14, Proposition 2.1] F is homeomorphic to the compact set $[0, 1] \times \{0, 1\}$. Thus, for each n , G_n can be taken to be the union of finitely many open sets. Every element y that lies between the upper and lower bound of F , has a neighbourhood of the form of equation (**). Thus, each G_n contains all but finitely many elements of the set $I \times \{\frac{1}{2}\}$. Since there are only countably many $n \in \mathbb{Z}_{++}$, the deletion of a countable union of a finite number of points is still countable, so the intersection $\bigcap_1^\infty G_n$ contains elements of $I \times \{\frac{1}{2}\}$. \square

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